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THE SEPARATRIX OF AN UNSTABLE POSITION OF EQUILIBRIUM OF A HESS–APPELROT GYROSCOPE†

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The motion of a heavy solid with a fixed point whose inertial tensor and the centre of mass (which does not coincide with the point of support) satisfy the Hess–Appelrot (HA) conditions is considered. At the zeroth value of the areas constant, the gyroscope has an unstable position of equilibrium at which the radius vector drawn from the point of support to the centre of mass is directed vertically upwards. Solutions which are asymptotic to this position of equilibrium form two-dimensional ingoing and outgoing separatrices which satisfy the Hess conditions and are therefore identical (they are paired). The motion close to a paired separatrix is considered (when, generally speaking, the particular Hess integral may be non-zero) and families of long-period solutions are found. Splitting of the separatrices when an HA gyroscope is perturbed is studied. The results obtained are used to investigate the separatrices of a perturbed Lagrange problem for a value of the areas constant close to zero. In particular, the occurrence of double-detour homoclinic solutions, which leads to the non-integrability of the problem, is demonstrated in the case of a zero value for the areas constant. The occurrence of single-detour homoclinic solutions of the perturbed Lagrange problem, leading to non-integrability for non-zero values of the areas constant has previously been found in [1].

1. FORMULATION OF THE PROBLEM

IN A NUMBER of cases it is convenient to make use of a special system of coordinates [2, 3] in the study of a solid with a fixed point, that is, a Cartesian system of coordinates which is rigidly fixed in the body where the unit vector directed from the point of support to the centre of gravity has the

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form $\mathbf{r}_0 = (1, 0, 0)$ and the kinetic energy T is expressed in terms of the components of the vector $\mathbf{G} = (x, y, z)$ of the kinetic moment using the formula

$$2T = (A\mathbf{G}, \mathbf{G}) = ax^2 + a_1y^2 + a_2z^2 + 2x(b_1y + b_2z)$$
$$A = (a_{ij}) = \begin{Vmatrix} a & b_1 & b_2 \\ b_1 & a_1 & 0 \\ b_2 & 0 & a_2 \end{Vmatrix} \quad (i, j = 0, 1, 2)$$

(A is the corresponding gyrational tensor). The equations of motion have the form

$$\mathbf{G}^{\cdot} = \mathbf{G} \times \omega + \Gamma \mathbf{r}_0 \times \gamma, \quad \gamma^{\cdot} = \gamma \times \omega \tag{1.1}$$

and possess the following integrals: the geometric integral $(\gamma, \gamma) = 1$, the areas integral $(\mathbf{G}, \gamma) = j$ and the energy integral $E = T - \Gamma\gamma_0$. Here, $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ is a unit vector which indicates the direction of the force of gravity (vertically downwards), $\omega = A\mathbf{G} = (\omega_0, \omega_1, \omega_2) = (p, q, r)$ is an angular velocity vector and Γ is the product of the weight of the body by the distance from the fixed point up to its centre of gravity.

An HA gyroscope is characterized by the conditions $a_1 = a_2 = a^*$. By rotating the special system of coordinates around the first axis it is possible to satisfy the conditions $b_2 = 0$, $b_1 \geqslant 0$ [3]. The corresponding equations of motion admit of the particular Hess integral $x = 0$. It is convenient to put $c = 2b_1/a^*$ and to change to the dimensionless variables

$$\begin{aligned} &\omega_i = 2\sqrt{b_1\Gamma}\,\omega_i'/c, \quad \mathbf{G} = \mathbf{G}'\sqrt{\Gamma/b_1} \\ &j = j'\sqrt{\Gamma/b_1}, \quad t = t'/\sqrt{b_1\Gamma}, \quad E = \Gamma h \end{aligned} \tag{1.2}$$

after which the equations take the form [3]

$$\begin{aligned} &\rho\rho^{\cdot} = \sqrt{f(\rho)}, \quad \rho^2\varphi^{\cdot} = -\rho^3\cos\varphi + j', \quad f(\rho) = \rho^2(1 - \gamma_0^2) - j'^2 \\ &\gamma_0 = c^{-1}\rho^2 - h, \quad \gamma_1 = j'\rho^{-1}\cos\varphi + \rho^{\cdot}\sin\varphi, \quad \gamma_2 = j'\rho^{-1}\sin\varphi - \rho^{\cdot}\cos\varphi \end{aligned} \tag{1.3}$$

in polar coordinates $\omega_1' = \rho\cos\varphi$, $\omega_2' = \rho\sin\varphi$.

In the general case ρ^2 is an elliptic function of time which varies between the two non-negative roots of the polynomial $P(u) = f(\sqrt{u})$. It follows from (1.3) that the geometric, areas and energy integrals and the particular Hess integral are independent everywhere at a two-dimensional common level in six-dimensional phase space apart from in the case of precessional motions ($\gamma_0 = \text{const}$) which correspond to the case when ρ^2 is identically equal to the multiple root of the polynomial P. The corresponding constraint on the integrals has the form $h = 1$, $j = 0$ or [3] $2c[h^3 - 9h - (h^2 + 3)^{3/2}] + 27j'^2 = 0$. In the first case, all of the motions are doubly asymptotic to the unstable position of equilibrium and this situation is discussed below. In the second case, the range of variation of ρ and, correspondingly, the common level of the integrals, degenerate into a single point and trajectory of the precessional motion.

At a zero value of the areas constant j and the critical value $h = 1$ of constant energy, the solution of the first equation (1.3) has the form

$$\rho = \sqrt{2c}/\operatorname{ch}\tau, \quad \tau = \sqrt{2/c}\,t' + t_0 = \sqrt{a^*}\,\Gamma t + t_0, \quad t_0 = \text{const}$$

The second equation (1.3) is solved by the method of separation of variables (since $j = 0$):

$$\sin\varphi = \operatorname{th}g, \quad \cos\varphi = \epsilon/\operatorname{ch}g$$

Here and henceforth, we shall use the notation $g = C - 2c\operatorname{arctg}e^{\tau}$, where C is a real constant and $\epsilon = \pm 1$. The solutions which have been found are doubly asymptotic to the position of equilibrium and, in the dimensionless variables ω_i', $\mathbf{G}' = (x', y', z')$, they have the form

$$y' = \omega_1' = \frac{\epsilon\sqrt{2c}}{\operatorname{ch}\tau\operatorname{ch}g}, \quad z' = \omega_2' = \frac{\sqrt{2c}}{\operatorname{ch}\tau}\operatorname{th}g$$

$$\gamma_0 = \frac{2}{\operatorname{ch}^2\tau} - 1, \quad \gamma_1 = -2\frac{\operatorname{sh}\tau}{\operatorname{ch}^2\tau}\operatorname{th}g, \quad \gamma_2 = 2\epsilon\frac{\operatorname{sh}\tau}{\operatorname{ch}^2\tau\operatorname{ch}g} \tag{1.4}$$

On passing from the dimensionless variables y' and z' to the initial variables y and z, the coefficient $\sqrt{2c}$ in (1.4) is replaced by $2\sqrt{\Gamma/a^*}$. When $c=0$, we have the well-known doubly asymptotic planar motions of a Lagrange gyroscope (the motions of a mathematical pendulum).

Hence, the solutions (1.4) are numbered with the sets (C, ϵ). We note that the pairs of cases $C=-\infty$, $\epsilon=\pm1$ and $C=+\infty$, $\epsilon=\pm1$ are identical and the corresponding solutions are planar (pendulum-type) solutions: $y\equiv0$, $\gamma_3\equiv0$. The fact that the gyroscopic motions, which are asymptotic to the equilibrium position, are Hessian was apparently pointed out for the first time by Appelrot [4, p. 130].

Let us now consider the rotation of an asymmetric Euler–Poinsot gyroscope around the central axis of inertia. Then, the eigenvalues of the corresponding monodromy matrix of the mapping after a period, which differ from unity, have the form $\exp(\pm2\pi\Lambda)$, where $\Lambda>0$ is a certain quantity [5, 6]. It can be shown that in the limit $\Lambda = c/2 = b_1/a^*$ if one considers rapid rotations of an HA gyroscope with $c>0$ (in the case when $c=0$, an HA gyroscope is a Lagrange gyroscope and, in the case when $c>0$, it has an asymmetric ellipsoid of inertia). The set of all possible values of Λ forms [7] the interval (0; 1) and, as is well known, in the case of a body with a specified asymmetric inertial tensor, it is possible to choose the position of the centre of mass which differs from the point of support in such a manner that the Hess condition is satisfied. The set of possible values of c is therefore [0; 2). (It has previously been shown in [8] that $c\leqslant2$ but there was no discussion as to which of the above-mentioned values may be taken.) If, however, the body contains an ellipsoidal cavity filled with an incompressible ideal fluid which executes non-vortex motions, then any values of $\Lambda>0$ and, correspondingly of $c>0$, may be taken [9].

We note that, according to the geometric picture of the motion given by Zhukovskii (see [4, 10]), the centre of mass of an HA gyroscope moves as a certain spherical pendulum with the same values of the energy and area integrals and with the same kinetic moment. In particular, when $j=0$, this pendulum executes planar motions.

2. CONSTRUCTION OF THE RECURRENCE MAPPING FOR MOTIONS CLOSE TO THE SEPARATRIX

Close to the position of equilibrium O: $\mathbf{G}=0$, $\boldsymbol{\gamma}=(-1,0,0)$ of the system (1.1) it is possible to choose y, z, γ_1 and γ_2 as local coordinates at a level $M^4\subset\mathbf{R}^6$ of the geometric and $j=0$ area integrals. According to (1.2) and (1.4), the relationships (only the upper or lower signs are taken)

$$y \sim 4me^{\pm\tau}\cos\varphi_\mp, \quad \gamma_1 \sim \pm4e^{\pm\tau}\sin\varphi_\mp$$

$$z \sim 4me^{\pm\tau}\sin\varphi_\mp, \quad \gamma_2 \sim \mp4e^{\pm\tau}\cos\varphi_\mp, \quad t\to\mp\infty$$

hold in the case of solutions which are asymptotic to the point O, where $m=\sqrt{\Gamma/a^*}$ and $\varphi_\pm$ are the limiting values of the angles φ:

$$\sin\varphi_- = \operatorname{th}C, \quad \cos\varphi_- = \epsilon/\operatorname{ch}C$$

$$\sin\varphi_+ = \operatorname{th}(C-c\pi), \quad \cos\varphi_+ = \epsilon/\operatorname{ch}(C-c\pi)$$

Hence, in the linear approximation, the outgoing separatrix W^- is defined by the conditions $X_+=Y_+=0$ and the ingoing separatrix W^+ is defined by the conditions $X_-=Y_-=0$, where the new local coordinates are related to the old coordinates by the relationships

$$y = 2m(X_- + X_+), \quad \gamma_1 = 2(Y_- - Y_+)$$

$$z = 2m(Y_- + Y_+), \quad \gamma_2 = 2(X_+ - X_\pm)$$

and, moreover, for asymptotic solutions in the linear approximation

$$X_{\mp}=2e^{\pm\tau}\cos\varphi_{\mp},\quad Y_{\mp}=2e^{\pm\tau}\sin\varphi_{\mp},\quad t\to\mp\infty \tag{2.1}$$

The reduced system in M^4 is Hamiltonian and the corresponding Poisson brackets have the standard form [11]. Calculations show that

$$\{X_-,Y_-\}=\{X_-,Y_+\}=\{Y_-,X_+\}=\{X_+,Y_+\}=0$$

$$\{X_-,X_+\}=\{Y_-,Y_+\}=1/(8m)$$

at the point O. It is therefore possible to introduce the canonically conjugate variables $P=(p_1,p_2)$, $Q=(q_1,q_2)$ such that

$$(p_1,p_2)=2\sqrt{2m}(X_+,Y_+),\quad (q_1,q_2)=2\sqrt{2m}(X_-,Y_-)$$

at the point O in the linear approximation.

Next, the characteristic exponents of the linearized system at the point O form two pairs of identical numbers $\pm\sqrt{a^*\Gamma}$, that is, there is a second-order resonance. By carrying out successive steps of the Birkhoff normalization procedure, it is possible to remove the non-resonant terms in the Hamiltonian H of the reduced system.

Let us introduce the notation $\|P\|=\sqrt{p_1^2+p_2^2}$, $\|Q\|=\sqrt{q_1^2+q_2^2}$ and note that terms of the orders $\|P\|\|Q\|^r$, $\|Q\|\|P\|^r$, where $r\geqslant 2$, are far from resonance terms. Their subsequent annihilation does not lead to the appearance of small denominators and the corresponding procedure of partial normalization converges (compare with [12]). Thus, in the neighbourhood of a point $O\in M^4$, canonically conjugate variables exist at which the Hamiltonian (a constant of the energy) has the form

$$H=k^{-1}(p_1q_1+p_2q_2)+O(\|P\|^2\|Q\|^2),\quad k^{-1}=\sqrt{a^*\Gamma}$$

(apart from an unimportant constant term) and the separatrices are defined by the conditions:

$$W^-:P=0;\quad W^+:Q=0$$

Let $s=p_2q_1-p_1q_2$.

As local coordinates which are tranverse to the two-dimensional separatrix, it is convenient to take s (or x) and H. We have

$$x=y\gamma_1+z\gamma_2=8m(X_+Y_--X_-Y_+)=-s$$

apart from higher-order terms. Next, in the approximation which is linear with respect to s and H, Q is expressed in terms of P, s and H using the formula

$$(q_1,q_2)=\|P\|^{-2}(s(p_2,-p_1)+kH(p_1,p_2)) \tag{2.2}$$

and P is expressed in terms of Q, $s'=s$ and H using the analogous formula

$$(p_1,p_2)=\|Q\|^{-2}(-s'(q_2,-q_1)+kH(q_1,q_2))$$

If $\omega=\|P\|\|Q\|$, then $\omega^2=s^2+(kH)^2$, apart from higher-order terms in s and H. The quantity ω characterizes the distance to the separatrix at the level of the integrals, M^4. Let us also introduce the angles θ and θ':

$$(p_1,p_2)=\|P\|(\cos\theta,\sin\theta),\quad (q_1,q_2)=\|Q\|(\cos\theta',\sin\theta') \tag{2.3}$$

It follows from (2.2) that

$$\theta'=\begin{cases}\theta-\operatorname{arc\,tg}[s/(kH)], & H>0\\ \theta-\tfrac{1}{2}\pi\operatorname{sign}s, & H=0\\ \theta+\pi-\operatorname{arc\,tg}[s/(kH)], & H<0\end{cases} \tag{2.4}$$

Furthermore, the magnitudes of ω, s, θ and θ' only vary slightly during motion close to the hyperbolic point O. In fact, it is seen that the terms $O(\omega^2)$ in the Hamiltonian H have a small effect on the change in these quantities (which remain constant in the corresponding linearized system). More accurately speaking, the relationships

$$|s^{\cdot}| = O(\omega^2), \quad |\omega^{\cdot}| = O(\omega^2), \quad |\theta^{\cdot}| = O(\omega), \quad |\theta'^{\cdot}| = O(\omega)$$

hold in a certain complex neighbourhood of the point O (here s, ω, θ and θ' are already complex quantities). Here and subsequently, a dot denotes differentiation with respect to dimensional time t. Let us now introduce the real regular analytic three-dimensional areas $\Pi^{\pm}$ which transversally intersect all real solutions lying in $W^{\pm}$. It may be assumed that the area Π^{+} is defined by the condition $\|P\| = \text{const} > 0$ and Π^{-} is defined by the condition $\|Q\| = \text{const} > 0$. Then, a neighbourhood U of the set $g = W^{+} \cap \Pi^{+}$ is found in Π^{+} such that an analytic mapping S_0: $U\backslash g \to \Pi^{-}$ along the trajectories of the reduced system is defined.The corresponding time of the motion is $T = O(-\ln\omega)$ and the increments in the quantities being considered are

$$|\Delta s/\omega|, \quad |\Delta\omega/\omega|, \quad |\Delta\theta|, \quad |\Delta\theta'| < O(T\omega) = O(\omega\ln\omega)$$

These estimates remain valid if one considers segments of $W^{\pm}$, $\Pi^{\pm}$ and g which lie in a complex space close to the corresponding real objects. The real recurrence mapping S_0: $(H, s, \theta) \to (H, s', \theta')$ is defined by the approximate formulas $s' = s$ and (2.4) and the corresponding error turns out to be an analytic function which is $O(\omega\ln\omega)$-small in the sense of the following definition.

Definition. We shall say that an analytic function S: $(H, s, \theta) \to (H', s', \theta')$, defined in $U\backslash g$ with values in Π^{-} is $O[(f(\omega)]$-small if the magnitude of θ' is $O[f(\omega)]$-small and the quantities H' and s' are $O[\omega f(\omega)]$-small in a complex domain which is a C_1-neighbourhood with respect to the θ coordinate and a $C_1\omega$-neighbourhood with respect to the H and s coordinates of the real domain where ω does not change by a factor of more than C_2. Here C_1 and C_2 are certain positive constants. Hence, if the coordinates, s, s', H and H' are "stretched" such that the magnitude of ω becomes of the order of unity, the function S becomes $O[f(\omega)]$-small in a certain fixed complex domain (small in a C^{ω}-norm). In particular, all of its derivatives will be of the same order of smallness.

Terms of the orders $\|P\|^{\alpha}\|Q\|^{\beta}$, where $\alpha/\beta \geq 2$ or $\beta/\alpha \geq 2$ far from resonance terms can be eliminated in the Hamiltonian H by means of a certain canonical substitution. The series, defining H, then converges at fairly small values of the product $\omega = \|P\|\|Q\|$ even if one of the factors is not small. By using this fact, it is possible to extend the coordinates P and Q into the neighbourhood V of a paired separatrix W such that the closure of V does not contain the point O. The above-mentioned extension can be carried out using two methods: along the separatrix W^{-} (the coordinates P^{-}, Q^{-}) and along the separatrix W^{+} (the coordinates P^{+}, Q^{+}).

Let us find the corresponding transition formulas in the domain V (compare with [7, 9]).

It follows from (2.1) that, in the linear approximation $\|Q^{-}\| = 2\sqrt{8me^{\tau}}$, $t \to -\infty$ and $\|P^{+}\| = 2\sqrt{me^{-\tau}}$, $t \to +\infty$ in the case of doubly asymptotic solutions. Here, since $P^{+} = -k^{-1}P^{+}$ and $Q^{-\cdot} = k^{-1}Q^{-}$ on the separatrix W, the identity $\|P^{+}\|\|Q^{-}\| = 32m$ is satisfied everywhere in W. Furthermore, in the doubly asymptotic solutions, the differential dx satisfies the equation in the variations $d(dx)/dt' = -zdx$, and ds is conserved. In this case $dx \to (dx)_{\pm}$ for $t \to \pm\infty$, where $dx_{+} = dx_{-}\mathrm{ch}(C - c\pi)/\mathrm{ch}C$. On the other hand, $x = -s$ is satisfied at the hyperbolic point O with an error in terms of higher than the second order and, hence, $ds' = -dx_{-}$, $ds = -dx_{+}$, where the quantities s' and s $(= p_2q_1 - p_1q_2)$ correspond to the systems of coordinates P^{-}, Q^{-} and P^{+}, Q^{+}. For each solution lying on the separatrix W, the angles θ and θ', which are related to P^{+} and Q^{-} by formulas (2.3), are constant quantities. Hence $\theta = \varphi_{+}$ and $\theta' = \varphi_{-}$.

With an $O(\omega)$-small error in the case of the given definition, the transition formulas S_1: $(H, s', \theta') \to (H, s, \theta)$ have the form

$$s = [\mathrm{ch}(C - c\pi/\mathrm{ch}C]\ s'$$

$$\cos\theta = \epsilon/\mathrm{ch}(C - c\pi), \quad \sin\theta = \mathrm{th}(C - c\pi)$$

where

$$\cos\theta' = \epsilon/\mathrm{ch}C, \quad \sin\theta' = \mathrm{th}C \tag{2.5}$$

(that is, terms independent of H and s' are left in the expression for θ while, correspondingly, terms which are linear in H and s' are retained in the expression for s) or, in the coordinates C, s, where C is chosen according to (2.5) and the primes are omitted,

$$C \to C - c\pi, \quad s\cos\theta = \text{const} \tag{2.6}$$

It is seen that the invariant Liouville measure of the Hamiltonian flow of the reduced system generates an invariant measure of the recurrence mapping on a three-dimensional surface which, when $s = 0$ and $H = 0$, has the form $ds(s')\wedge dH\wedge D\theta(\theta')$. It can be verified that the mapping S_1 in the approximation being considered retains this 3-form. Actually, it follows from (2.5) that $d\theta' = \cos\theta' dC$ and, hence, $d(s'\cos\theta')\wedge dC = ds'\wedge d\theta'$, after which it is convenient to use formula (2.6). An analogous result is apparently also valid for the mapping S_0.

In order to study motions close to the separatrix W it is advisable to take the three-dimensional area Π^+ or Π^- and to consider the corresponding recurrence mapping $S = S_1 \circ S_0$ or $S = S_0 \circ S_1$ which will be two-dimensional when account is taken of the presence of the integral H. If it is assumed that the area is defined by the condition $\|P\| = \text{const}$ or $\|Q\| = \text{const}$, then the corresponding time of the motion is $k\ln(32m/\omega)$ with an error of $O(\omega\ln\omega)$ in the sense of the definition given earlier.

We note that all of the estimates $O(\omega)$ and $O(\omega\ln\omega)$ will be uniform on any compactum from the space of the parameters of the problem

$$\{(a, a^*, b_1)\in \mathbf{R}^3:\ a>0,\ a^*>0,\ b_1\in\mathbf{R},\ b_1^2 < aa^*\}$$

3. PERIODIC MOTIONS CLOSE TO THE SEPARATRIX

As applications of the results in Sec. 2, let us consider the questions of periodic solution close to the separatrix of an HA gyroscope for which $c>0$. In order to do this, it is necessary to find the periodic units of the recurrence mapping S. The periodic point of the mapping S which has the smallest period n corresponds to an n-detour periodic solution, that is, a solution which circumvents the separatrix n times. All quantities are subsequently indicated with an error $O(\omega\ln\omega)$.

1. When $s = 0$ and $H>0$, the mapping S only has two fixed points $\theta = \theta' = \pm\pi/2$. They correspond to planar rotations which exist under Hessian conditions by virtue of the symmetry of the problem. Simple calculations show that the eigenvalues of the mapping S at the fixed points are $\exp(\pm c\pi)$. Consequently, the planar rotations are hyperbolic. It is seen from an analysis of the change in the angle $\theta = \theta'$ during the iterations of the mapping S that the two-dimensional regular surface $M_H^2\subset M^4$ which is cut out by the energy integral H and the particular Hessian integral coincides with the outgoing sepatrices of one of these hyperbolic trajectories and with the ingoing separatrices of the second. An analogous result for any $H>0$ follows from the second equation of (1.3) since $\rho\geqslant\rho_0(H)>0$ always.

2. When $s = 0$ and $H<0$, the mapping S does not have points with an odd period (for sufficiently small $-H$). The formulas obtained in Sec. 2 do not suffice to find the points of even period. Actually, it follows from these formulas that the entire circumference $s = 0$ consists of the fixed degenerate points of the mapping S^2. This picture can be destroyed by the terms $O(\omega\ln\omega)$ which have not been taken into account. In particular, to a planar vibration (a librational motion) of the gyroscope there correspond the fixed points of the mapping S^2, which are $O(H)$-close in the sense of the definition in Sec. 2 to $s = 0$, $\theta = -\theta' = \pm\pi/2$, while the eigenvalues are real eigenvalues which are $O(H\ln(-H))$-close to unity. By making use of symmetry considerations, it can be shown that, for all $0>H = E-\Gamma>-2\Gamma$, the surface M_H^2 is filled with closed degenerate trajectories. The mapping S therefore actually has a closed curve of degenerate fixed points which are $O(H)$-close to the circumference $s = 0$. It is unclear, however, whether there are other fixed points in the $O(H\ln H)$-neighbourhood of this circumference.

3. When $s\neq 0$ and $H>0$, the mapping S has two fixed points $\theta = -\theta' = \tfrac{1}{2}\operatorname{arctg}[s/(kH)] \bmod \pi$ and it can be found that

$$s/(kH) = -2\epsilon\,\mathrm{sh}C/(1-\mathrm{sh}^2 C);\quad C = c\pi/2,\quad \mathrm{tg}\,\theta = -\epsilon\,\mathrm{sh}C \tag{3.1}$$

and the constant $\mathrm{sh}C<1$ holds.

4. When $s\neq 0$ and $H<0$, the mapping S has two fixed points

$$\theta = -\theta' = \tfrac{1}{2}(\mathrm{arc\,tg}[s/(kH)] - \pi) \bmod \pi$$

formulas (3.1) hold and, moreover, the constraint $\mathrm{sh}C>1$ holds.

5. In the limiting version for cases 3 and 4 we have $\mathrm{sh}C = 1$, $H = 0$, $\theta' = -\theta = \pi/4 \bmod \pi$ or $3\pi/4 \bmod \pi$, respectively, when $s<0$ or $s>0$. Then, as in case 2, we have whole lines consisting of degenerate fixed points of the mapping S with terms $O(\omega\ln\omega)$ which have not been taken into account.

Simple calculations show that the trace of the linear part of the mapping S at the fixed points for cases 3, 4 and 5 is

$$A(v) = 2[1 - 2v(1 - v)/(1 + v)^2], \quad v = \mathrm{sh}^2 C.$$

The next assertion follows from the arguments which have been presented.

Theorem 1. 1. All of the single-detour periodic solutions which are sufficiently close to the separatrix are described by cases 1, 3 and 4 which excludes the obscure situation of case 5 for $\mathrm{sh}\, C = 1$.

2. Solutions which are described by cases 3 and 4 are elliptic when $\mathrm{sh}\, C < 1$, that is, $c < c_* = 2\pi^{-1}\ln(1+\sqrt{2}) \approx 0.561$ (case 3) or hyperbolic when $\mathrm{sh}\, C > 1$, that is $c > c_*$ (case 4).

Remark. The function $A(v)$ decreases when $v < 1/\sqrt{3}$, that is $c < c_{**} = 2\pi^{-1}[\ln(1+\sqrt{1+\sqrt{3}}) - \frac{1}{4}\ln 3]) \approx 0.446$ and increases when $v > 1/\sqrt{3}$, that is, when $c > c_{**}$. The minimum value of $A(v)$ is $A(1/\sqrt{3}) = 6(2-\sqrt{3}) \approx 1.61$; $A(v) \to 6$ when $v \to +\infty$.

4. SPLITTING OF THE SEPARATRICES OF THE PERTURBED PROBLEM

Using first-order perturbation theory, let us study the splitting of the separatrices of an unstable permanent rotation if the HA conditions are violated. By virtue of the symmetry, it is sufficient to consider perturbations under which just one of the following three quantities varies: a_{22} (or a_{11}), a_{12}, j. Here $A = (a_{ij})$ is a gyrational tensor which corresponds to a moving system, the first axis of which is directed from the point of support to the centre of gravity. It is convenient to introduce the parameter

$$\delta = [(a_{11} - a_{22})^2 + a_{12}^2 + (dj)^2]^{1/2}$$

(the arbitrary factor d which has the dimensions of "moment of inertia$^{-2}\times$time" serves to reduce the terms to a single dimension), which characterizes the difference between the gyroscope which is being considered and an HA gyroscope.

In order to study the behaviour of the solutions of the reduced system (1.1) as a function of the value of the constant area j, it is convenient to proceed in the following manner. Let us put $\mathbf{M} = \mathbf{G} - j\boldsymbol{\gamma}$. Then, $(\mathbf{M}, \boldsymbol{\gamma}) = 0$ and, after replacing $\mathbf{G}$ by $\mathbf{M}$, Eqs (1.1) preserve their form but the angular velocity will be calculated using the formula $\boldsymbol{\omega} = A\mathbf{M} + jA\boldsymbol{\gamma}$. Hence, the case when $j \neq 0$ reduces to the case when $j = 0$ with a change in the formula for the dependence of $\boldsymbol{\omega}$ on $\mathbf{M}, \boldsymbol{\gamma}$. Henceforth, $\mathbf{M} = (x, y, z)$.

So, there are three independent perturbation parameters $\alpha = a_{22} - a_{11}$, $\beta = a_{12}$ and j.

It is convenient to use the well-known technique [13, 14] in order to study the splitting of the separatrices. Let W be the unperturbed (binary) separatrix and W^+ and W^- be the perturbed ingoing and outgoing separatrices, respectively. Let $U \subset \mathbf{R}^6\{\mathbf{M}, \gamma\}$ be a certain domain such that: (1) the closure $\bar{U}$ does not contain the unperturbed fixed point of the reduced system (1.1), and (2) the boundary ∂U transversally intersects the unperturbed separatrix W. There then exists an analytic retraction π: $U \to W$ (that is, a mapping which is identical in the set $U \cap W$). The unperturbed separatrix W is singled out by the three classical integrals of the problem and the particular Hess integral $x = 0$. We match points $w^\pm \in W^\pm$ to each point $w \in U \cap W$ such that $\pi(w^\pm) = w$. Let $x_\pm(w)$ be the x-coordinate of a point $w^\pm \in \mathbf{R}^6$.

In order to study the splitting of the separatrices $W^\pm$, it is necessary to calculate the first-order terms in the expansion of the Mel'nikov function $\Delta(w) = x(w) - x_+(w)$ with respect to the small perturbation parameters α, β and j. For this purpose, it is sufficient to know the formulas of the unperturbed doubly asymptotic solutions (1.2), (1.4) and the corresponding differential equation for the change in x:

$$x^{\cdot} = -b_1 xz + \alpha yz + \beta(y^2 - z^2) + j(a^*(y\gamma_2 - z\gamma_1) - b_1 z\gamma_0) + \ldots$$

(terms of higher order with respect to α, β and j are denoted by the string of dots). After transforming to dimensionless variables using (1.2) and $\alpha = b_1\alpha'$, $\beta = b_1\beta'$, this equation has the form (the primes are omitted here and henceforth)

$$dx/dt = x^{\cdot} = -zx + \alpha yz + \beta(y^2 - z^2) + j\left(\frac{2}{c}(y\gamma_2 - z\gamma_1) - z\gamma_0\right) + \ldots \tag{4.1}$$

The functions $x_\pm(w)$ are analytically dependent on the small α, β and j and, to calculate the first-order terms, it is advisable to proceed in the following manner. Let $v(t)$ be a doubly asymptotic solution (1.4) such that

$v(T) = w$ for a certain T and $v_\pm(t)$ is a solution, lying on $W^\pm$, which is asymptotic when $t \to \pm\infty$, respectively, and such that $v_\pm(T) = w^\pm$; $x_\pm(t)$ is the x-coordinate of a point $v_\pm(t)$ which satisfies (4.1). In order to find the first-order terms in the expansion of $x_\pm(t)$, it is sufficient to solve Eq. (4.1) under the assumption that the values of y, z, γ_0, γ_1 and γ_2 correspond to the unperturbed solution $v(t)$ and the boundary condition $x_\pm(t) \to 0$, $t \to \pm\infty$, which corresponds to the fixed point of the reduced system, is imposed. Equation (4.1) will then have the form $x^{\cdot} = f(t)x + h(t)$, where $f(t) = -z(t)$ and $h(t)$ are known functions of time. A simple calculation shows that $-zdt = d(\ln \mathrm{ch} g)$.

By using the method of undetermined multipliers, we obtain the solution in the form

$$x_\pm(t) = \mathrm{ch}\, g \int_{\pm\infty}^{t} \frac{h(t_1)}{\mathrm{ch}\, g(t_1)}\, dt_1$$

$[g(t_1)$ is the value of the function g calculated under the assumption that $t = t_1]$ and, consequently

$$\Delta(w) = x_-(T) - x_+(T) = \mathrm{ch} g(T) I$$

apart from higher-order terms, where

$$I = \int_{-\infty}^{+\infty} \frac{h(t)}{\mathrm{ch} g(t)}\, dt$$

By making the substitution $l = 2\,\mathrm{arctg}\, e^\tau$, it can be shown that

$$I = \alpha I_\alpha + \beta I_\beta + j I_j$$

$$I_\alpha = \epsilon c \sqrt{2c}\, J_\alpha, \quad I_\beta = c\sqrt{2c}\, J_\beta, \quad I_j = J_j$$

$$J_\alpha = \int_0^\pi \sin l \,\frac{\mathrm{sh} g}{\mathrm{ch}^3 g}\, dl, \quad J_\beta = \int_0^\pi \sin l \left(\frac{2}{\mathrm{ch}^3 g} - \frac{1}{\mathrm{ch} g}\right) dl$$

$$J_j = \int_0^\pi \left(-2\,\frac{\sin 2l}{\mathrm{ch} g} + c\cos 2l\, \frac{\mathrm{sh} g}{\mathrm{ch}^2 g}\right) dl, \quad g = C - cl$$

On returning to the initial (dimensional) quantities, we obtain

$$I_\alpha = 4\epsilon \sqrt{\Gamma/a^{*3}}\, J_\alpha, \quad I_\beta = 4\sqrt{\Gamma/a^{*3}}\, J_\beta, \quad I_j = J_j$$

Without loss of generality, we shall assume that the perturbed Hamiltonian vanishes at the fixed point. Next, the perturbed Hamiltonian can be reduced to the form $H = H_2 + O(\omega^2)$ by means of a certain canonical substitution which is analytic with respect to the perturbation parameters α, β and j and identical when $\alpha = \beta = j = 0$, where the quadratic terms H_2 are of the order of $O(\omega)$. (This is possible due to the fact that, after reducing H_2 to the proper form, the terms $\|P\|^r\|W\|$, $\|P\|\|Q\|^r$, $r \geqslant 2$ remain remote from resonances.) Let s and s' be the variables which have been "rectified" in accordance with this substitution. The separatrices W^+ and W^- then satisfy the conditions $s = 0$ and $s' = 0$ respectively. Let $s_+(w)$ be the s-coordinate of the point w^- and $s_-(w)$ be the s'-coordinate of the point w^+. It follows from the results in Sec. 2 that, in the case of the unperturbed system, the relationships

$$-dx = A_- ds' = A_+ ds, \quad A_\pm = \mathrm{ch} g(T)/\mathrm{ch} g(\pm\infty)$$

are satisfied on the separatrix W.

The expansions of the functions $s_\pm(w)$ in series over α, β and j therefore hold and, in the linear approximation,

$$s_\pm(w) = \mp I \mathrm{ch} g(\pm\infty) \tag{4.2}$$

The following general result also follows from this. In the linear approximation with respect to α, β, j, s and H, the second of the transition formulas (2.6), $S_1^\sim$: $(H, s', \theta') \to (H, s, \theta)$ (here $S_1^\sim$ is a "perturbed" mapping S_1) takes the following form in the domain V:

$$s\cos\theta = -\epsilon I + s'\cos\theta'. \tag{4.3}$$

When there are no perturbations, $I \equiv 0$ and (4.3) reduces to (2.6) and, when $H = 0$ and $s' = 0$ or $s = 0$, (4.2) follows from (4.3). These results will be used in Sec. 5.

Theorem 2. 1. Let $c > 0$. Then, for sufficiently small δ, the splitting of the separatrices W^+ and W^-

is of the order of δ (see [9]) and this estimate is uniform on any compactum in the space of parameters which correspond to HA gyroscopes with $c>0$.

2. Let $c = 0$, that is, the unperturbed problem is the Lagrange case. Then, for sufficiently small δ, $|c|$, the splitting of the separatrices W^+ and W^- is of the order of $\max\{|\alpha|, |\beta|, |djc|\} = \delta_1$, and this estimate is uniform on any compactum in the space of parameters which correspond to the Lagrange case.

Proof. Point 1 immediately follows from the following assertion.

Lemma. When $c>0$, the quantities I_α, I_β and I_j, as functions from the set (C, ϵ), are linearly independent, that is $I = \alpha I_\alpha + \beta I_\beta + jI_j = 0$ for all C, ϵ only when $\alpha = \beta = j = 0$.

In order to show that this is so, it is sufficient to make use of the asymptotic formulas

$$\operatorname{ch} g \sim \tfrac{1}{2}\exp|g|, \quad \operatorname{sh} g \sim \tfrac{1}{2}\operatorname{sign} g \exp|g|, \quad g \to \pm\infty$$

and to carry out simple calculations.

In the case when $c = 0$, we have $I_j \equiv 0$ but the function $c^{-1}I_j|_{c=0}$ is defined correctly and it is not identically equal to zero. It can be shown that the lemma which has been formulated above holds if, instead of I_j and j, one takes $c^{-1}I_j|_{c=0}$ and jc, respectively. Next, in the expansion of $\Delta(w)$ in terms of α, β, j and c, there are no terms of the form j^r or c^r, since $\Delta \equiv 0$ in the HA case, when $\alpha = \beta = j = 0$, or in the Lagrange case, when $\alpha = \beta = c = 0$. Point 2 of the theorem follows from this.

Corollary. The separatrices W^+ and W^- are not split only when the perturbed problem is the Lagrange case or the areas constant is equal to zero and the HA conditions are satisfied.

We note that the splitting of the separatrices of the hyperbolic periodic solutions of a perturbed Euler–Poinsot problem (with an arbitrary value of the areas constant), which are obtained from permanent rotation around the central axis of inertia in the unperturbed problem, has been studied previously in [6, 5]. It was shown that the separatrices are always split, apart from in the HA case, when two pairs of separatrices remain double and coincide with the surface M_H^2 (compare with point 1 in Sec. 3) while two others are split.

Remark. 1. If the separatrices $W^\pm$ of a hyperbolic point of a Hamiltonian system with two degrees of freedom are split, they have at least two different lines of traverse intersection or tangency of odd order (compare with [1, 9]).

The proof is elementary and is based on the use of the Poincaré–Kartan integral invariant. In particular, this assertion is applicable to the reduced system (1.1) which can be represented in Hamiltonian form. In the case being considered, it also readily follows from the equality

$$\int_{-\infty}^{+\infty} I\,dC = 0 \quad \text{for } \epsilon = \pm 1$$

2. A simple calculation shows that, in the case of a perturbation of a Lagrange gyroscope,

$$\Delta = \tfrac{1}{2}\alpha \sin 2\varphi + \beta\cos 2\varphi + jc\pi \sin\varphi + \dots$$

where $\varphi = \varphi_- = \varphi_+$ is the angle which numbers doubly asymptotic trajectories. It can be shown that, for small $\delta_1 \neq 0$, there always exists a line of transverse intersection of the separatrices and even two such different lines if the parameters α, β, j and c do not lie in analytic manifolds with a codimensionality two, the equations of which, in the approximation which is linear in α, β and jc have the form $\beta = 0$, $\alpha \pm jc\pi = 0$.

5. APPLICATIONS TO THE PERTURBED LAGRANGE PROBLEM. ONE- AND TWO-DETOUR TRANSVERSAL HOMOCLINIC SOLUTIONS

In the case when $c = 0$, an HA gyroscope is a Lagrange gyroscope. Below, we will consider the perturbation of the Lagrange problem under which the centre of mass is shifted from the axis of dynamic symmetry in a perpendicular direction and the areas constant $j = 0$. As the perturbation parameter $\mu > 0$, we shall take the angle which is made by the axis of dynamic symmetry and a vector drawn from the fixed point to the centre of mass. Without any loss of generality, it may be assumed that the moments of inertia of the body are $A = B = 1$, $C \neq 1$. Then, in the proper special system of coordinates (the first two axes of which form the plane of symmetry of the gyroscope), the expressions

$$a = C^{-1} + K\mu^2 + O(\mu^4), \quad a_1 = 1 - K\mu^2 + O(\mu^4)$$
$$b_1 = K\mu + O(\mu^3), \quad b_2 = 0, \quad a_2 = 1; \quad K = C^{-1}(C-1)$$

hold for the components of the gyrational tensor where the quantities a and a_1 are even with respect to μ, and b_1 is odd. A perturbed Lagrange problem can therefore be considered as a perturbation of an HA gyroscope with the parameter $\alpha = a_2 - a_1 = K\mu^2 + O(\mu^4)$ for which

$$a^* = a_2 = 1, \quad c = 2b_1/a^* = 2K\mu + O(\mu^3).$$

In view of this, the splitting of the separatrices is of the second order of smallness with respect to μ which was also discovered in [1]. The Mel'nikov function $\Delta(w) = x_-(w) - x_+(w)$ is expanded in a series with respect to α with functional coefficients which depend on c. It is seen that

$$\Delta(w) = \alpha I_\alpha|_{c=0} + \ldots = \mu^2 f_0(\varphi) + \ldots, \quad f_0(\varphi) = 8K\sqrt{\Gamma}\sin\varphi\cos\varphi$$

with an error in the terms which are cubic in μ. Here, $\varphi = \varphi_- = \varphi_+$ is the angle which numbers doubly asymptotic trajectories of the Lagrange problem.

The function $f_0(\varphi)$ has four zeros and they are all simple. Hence, the perturbed separatrices $W^\pm$ have precisely four lines $L_i^{(1)}$ of transversal intersection. The so-called single detour homoclinic solutions (which pass round the unperturbed separatrix once) correspond to the lines. Of these four solutions, two are planar (the corresponding zeros of the function $f_0(\varphi)$ are $\varphi = \pm\pi/2$), that is, rotations around the third axis of inertia, while the two others are close to rotations around the second axis of inertia.

Let us now find the transversal homoclinic solutions $L_i^{(2)}$ which are two-detour solutions, that is, which pass around the unperturbed separatrix twice. According to Sec. 4, the lines $l_+ = W^- \cap \Pi^+$ and $l_- = W^+ \cap \Pi^-$ in the areas Π^+ and Π^- are, respectively, defined by the equations

$$l_+: H = 0, \ s = s_+ = f_+(\theta); \quad l_-: H = 0, \ s' = s_- = f_-(\theta')$$
$$f_\pm(\varphi_\pm) = \mp \alpha I_\alpha \epsilon/\cos\varphi_\pm + \ldots$$

where H is the unperturbed Hamiltonian and s and s' are the "rectified" variables. It can be found that

$$f_\pm(\theta) = \mu^2(\mp f_0(\theta) + \mu f_1(\theta) + O(\mu^2))$$
$$f_1(\theta) = 8\pi K^2\sqrt{\Gamma}(2\cos\theta - 3\cos^3\theta)$$

Following what has been said in Sec. 2, let us consider a neighbourhood U of the set $g = W^+ \cap \Pi^+$ in Π^+ and the mapping S_0^-: $U \backslash g \to \Pi^-$ along the trajectory of the perturbed system. A required line $L_i^{(2)}$ of transversal intersection of the separatrices W^+ and W^- corresponds to a point of transversal intersection of the curves $S_0^-(l_+)$ and l_-. The mapping S_0^- is completely analogous to S_0 which was constructed in Sec. 2. Let the constants $r_1 > 0$, $r_2 > 0$ and let the domain $D \subset U$ be defined by the inequalities $r_1\mu^2 < \omega < r_2\mu^2$. Then, the time of the motion from Π^+ to Π^- is $O(\ln\mu)$ for any point from D. We note that a system linearized at a fixed point O in the coordinates P and Q experiences a perturbation $O(\mu^2)$. Arguments, which are similar to those in Sec. 2, show that the mapping S_0^-: $(H, s, \theta) \to (H, s', \theta')$ in domain D has the form (2.4) and $s = s'$ with an $O(\omega\ln\omega) = O(\mu^2\ln\mu)$-small error (in the sense of the definition from Sec. 2). Then, when $H = 0$, we obtain

$$\theta' = \theta - \tfrac{1}{2}\pi \operatorname{sign} s + O(\mu^2\ln\mu) \tag{5.1}$$

In order to find the point of intersection $S_0^-(l_+) \cap l_-$, let us consider the equation

$$R(\theta) = s_+ - s_- = -f_-(\theta') + f_+(\theta) = 0 \tag{5.2}$$

where θ and θ' are related by the equality (5.1) and satisfy the conditions

$$|f_0(\theta)| > 2r_1, \quad |f_0(\theta')| > 2r_1, \quad r_1 > 0$$

The terms in $R(\theta)$ accompanying μ^2 then cancel out since $f_0(\theta) + F_0(\theta \pm \tfrac{1}{2}\pi) = 0$. We have $\mu^{-3}R(\theta) = -8\pi K^2\sqrt{\Gamma}(\epsilon\sin\theta + \cos\theta)(1 - 3\epsilon\cos\theta\sin\theta) + O(\mu\ln\mu)$, $\epsilon = \operatorname{sign} f_0(\theta)$.

When $K < 0$, that is, when $C < 1$, Eq. (5.2) has four simple zeros which are $O(\mu\ln\mu)$-close to the numbers θ which are such that $\operatorname{tg}\theta = \pm 1$.

When $K>0$, that is when $C>1$, Eq. (5.2) has eight simple zeros which are $O(\mu \ln \mu)$-close to the numbers θ which are such that $\sin 2\theta = \pm 2/3$.

Let V be the neighbourhood of the separatrix W, the boundary of which is a smooth three-dimensional manifold which is transverse to all of the doubly asymptotic solutions of the unperturbed problem. For small $\mu \neq 0$, the set $V \cap L_i^2$ has two connected components $L_i^{\pm}$ (L_i^- corresponds to a smaller value of time t).

Definition. We shall say that a two-detour homoclinic trajectory $L_i^{(2)}$ is of the type (θ_1, θ_2) if $L_i^- \to v^{\sim}(\theta_i)$, $L_i^+ \to v^{\sim}(\theta_2)$, $\mu \to 0$, where $v^{\sim}(\theta)$ is the trajectory of the unperturbed Lagrange problem when $j=0$, which corresponds to the doubly asymptotic solution for which $\varphi_- = \varphi_+ = \theta$.

Sheets of separatrices $W^{\pm}$, lying in the domain V close to the limiting point O (for the smallest or greatest possible value of the time t, respectively), have previously been considered. The separatrices $W^{\pm}$ may continue along trajectories of the system through a small neighbourhood of the point O. As a result, we obtain another set of sheets of separatrices $W_{\sim}^-$ and $W_{\sim}^+$. (These sheets will already be unconnected by virtue of the intersection of sheets $W^{\pm}$ of the "first order".) In particular

$$S_0^{\sim}(l_+) = W_{\sim}^- \cap \Pi^+, \quad L_i^- \subset W^- \cap W_{\sim}^+, \quad L_i^+ \subset W_{\sim}^- \cap W^+.$$

Theorem 3. 1. The separatrices $W^{\pm}$ of a perturbed Lagrange problem have four lines $L_i^{(1)}$ of tranverse intersection which are single-detour homoclinic trajectories. Two of them correspond to planar solutions and the other two correspond to solutions which, in the domain V, are $O(\mu)$-close to rotations around a horizontal axis lying in the plane of symmetry of the gyroscope. The splitting of the separatrices $W^{\pm}$ is of the order of $O(\mu^2)$ and, moreover, this estimate cannot be improved.

2. Transversal two-detour homoclinic trajectories $L_i^{(2)}$ of the following types exist:

1°. $(\theta, -\theta)$, where $\theta \in \{\pm\pi/4, \pm 3\pi/4\}$, if $C<1$.

2°. (θ_1, θ_2), (θ_2, θ_1), where $\theta_2 = \theta_1 + \pi/2$, $\theta_1 \in \{n\pi + l, n\pi + \pi/2 - l$, where $n = 0;\ 1\}$, $l = \tfrac{1}{2} \arcsin \tfrac{2}{3}$,

if $C>1$.

In the domain V, the connected components $L_i^{\pm}$ of $L_i^{(2)}$ will be $O(\mu \ln \mu)$-close to the corresponding doubly asymptotic trajectories $v^{\sim}(\theta)$ of the perturbed problem.

3. The sets $W_{\sim}^{\pm}$ lie in an $O(\mu^3)$-neighbourhood of $W^+ \cup W^-$. The set W^+ (W^- respectively) lies in an $O(\mu^3)$-neighbourhood of $W_{\sim}^-$ ($W_{\sim}^+$ respectively). These estimates cannot be improved upon for any small $\mu \neq 0$.

Proof. Points 1 and 2 follow directly from the arguments which have been presented above. In order to prove point 3, it is necessary to note that, first, it is possible to expand the domain D by assuming that it is defined by the conditions $r_1|\mu|^3 < \omega < r_2\mu^2$ and, second, that the mapping $S_0^{\sim}$ converts a point in Π^+, for which $|\omega| < r_1|\mu|^3$ into a point for which $|\omega| < 2r_1|\mu|^3$.

Using methods which have previously been proposed [1], it is possible, from the results of point 2 of Theorem 3, to infer the existence of quasi-random motions and the non-integrability of the perturbed Lagrange problem in the case of a zero value of the areas constant and a fixed small $\mu \neq 0$ at energy levels which are close to the critical level.

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EXTREMAL PROBLEMS OF HEAT TRANSFER TO THREE-DIMENSIONAL BODIES AT HYPERSONIC SPEEDS†

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The design of shuttle-like hypersonic spacecraft [70, 77] gives rise to the problem of investigating the spatial configurations that are optimum from the point of view of thermal heating and other characteristics, which enable the weight of the required thermal protection to be reduced. The problem of optimizing the weight of thermal protection depends on many parameters and has not yet been solved in a rigorous mathematical formulation. Approximate formulations of the optimization problem have therefore been considered, the solution of which has enabled axisymmetrical optimum shapes of bodies to be obtained with the minimum convective [43, 61, 73] and radiation [35–37, 58, 60] heat fluxes. It is known from attempts to solve variational problems of a body with minimum drag [29–31, 51–53, 62], that the transition to essentially three-dimensional configurations enables a reduction in the drag to be achieved compared with axisymmetrical bodies. A similar situation should obviously also occur when optimizing the shape of a body for heat flux. In this paper we present for the first time variational problems for finding the optimum shape of three-dimensional bodies of minimum overall thermal heating when moving along an incoming trajectory. In papers by other authors [21, 29–31, 48, 51–53, 62] the problems of determining the three-dimensional optimum aerodynamic shapes from the point of view of the minimum wave or total drag were considered. A brief review is given of research which has been done to determine the convective and radiation heating of three-dimensional bodies and the fundamental formulas for the wave drag, the fraction drag, and the convective and radiation fluxes to three-dimensional bodies moving in dense layers of planetary atmospheres are presented. The formulas depend explicitly on the conditions of entry into the atmosphere of the planet and on the geometry of the body, which enables variational problems to be formulated on determining the three-dimensional shape of the body from the conditions for minimum overall heating (convective and radiation) of the surface along the trajectory of motion.

1. FORMULATION OF VARIATIONAL PROBLEMS ON CHOOSING THE OPTIMUM SHAPE OF THREE-DIMENSIONAL BODIES OF MINIMUM OVERALL HEAT TRANSFER

CONSIDER the motion of a three-dimensional body in a planetary atmosphere along a plane trajectory at a hypersonic velocity acted upon by a lift force, a frontal drag, a gravity force and a

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